JOURNAL OF NUMBER THEORY 2, 106–115 (1970)

On the Fundamental Unit of Real Quadratic Fields with Norm 1

HIDEO YOKOI

Department of Mathematics, Nagoya University, Chikusa-ku, Nagoya, Japan

Communicated by H. Hasse

Received April 10, 1969

We give explicitly the fundamental unit of real quadratic fields of a new type different from Richaud-Degert's. Using this explicit form of the fundamental unit, we give an estimation from below of the class number of real quadratic fields. Finally, we add some tables of the fundamental unit of this type.

Throughout this paper, we shall understand by $\mathbf{Q}$ the rational number field, by $K = \mathbf{Q}(\sqrt{D})$ ($D > 0$ a rational square-free integer) a real quadratic field, and by $\epsilon_D = \frac{1}{2}(t_0 + u_0\sqrt{D})$ the normalized fundamental unit $\epsilon_D > 1$ of $K = \mathbf{Q}(\sqrt{D})$, i.e. $t_0 > 0$, $u_0 > 0$.

In our recent paper [4], we gave explicitly the fundamental unit of real quadratic fields of the more general type than Richaud-Degert's in the case of real quadratic fields with the fundamental unit ϵ satisfying $N\epsilon = -1$, and considered the class number of real quadratic fields of this type. In this paper, we consider the case of real quadratic fields with the fundamental unit ϵ satisfying $N\epsilon = 1$ and give explicitly the fundamental unit of real quadratic fields of a different type from Richaud-Degert's.

In Section 1, we classify all units $\epsilon = \frac{1}{2}(t + u\sqrt{D})$ with $N\epsilon = 1$ by the positive value of u, and prove that in the class of $u = p$ or $u = 2p$ (p is 1 or prime congruent to $-1 \bmod 4$) the unit $\epsilon = \frac{1}{2}(t + u\sqrt{D}) > 1$ becomes the fundamental unit of $\mathbf{Q}(\sqrt{D})$ except for at most finite number of values of D. Moreover, we give an explicit form of the fundamental unit (Theorem 1 and its Corollary). In Section 2, we give an estimation from below of the class number of real quadratic fields with the fundamental unit belonging to the class of $u = p$ or $2p$ (Theorem 2). Finally, we add some tables of the fundamental unit in the cases of $p = 3$, 7, and 11.

1. The Fundamental Unit

Before going into the main argument, we first prove the following proposition:

PROPOSITION 1. *Let p be any prime satisfying $p \equiv -1 \bmod 4$, and let $\epsilon_D = \frac{1}{2}(t_0 + u_0\sqrt{D})$ be the fundamental unit of a real quadratic field $\mathbf{Q}(\sqrt{D})$ ($D > 0$ square-free). Then, if we assume $u_0 \equiv 0 \bmod p$, $N\epsilon_D = 1$ holds and there exists a positive integer m such that $t_0 = p^2m \pm 2$, $(u_0/p)^2 D = p^2m^2 \pm 4m$.*

Proof. From the assumption $p \equiv -1 \pmod 4$ and $u_0 \equiv 0 \pmod p$ it follows that Pell's equation $x^2 - Dy^2 = -4$ is not solvable,[1] which implies $N\epsilon_D = 1$, i.e., $t_0^2 - Du_0^2 = 4$. Hence, we get $t_0^2 \equiv 4 \pmod{p^2}$, which implies $t_0 \equiv \pm 2 \pmod{p^2}$. Therefore, there exists a positive integer m such that $t_0 = p^2m \pm 2$. Since for this integer m we have $Du_0^2 = t_0^2 - 4 = p^2\ (p^2m^2 \pm 4m)$, $(u_0/p)^2 D = p^2m^2 \pm 4m$ holds.

Next, we shall prove that in the case of $u_0 = p$ (congruent to -1 mod 4) the inverse of this proposition is true, and give explicitly the fundamental unit of real quadratic fields of a new type. For this purpose we must prepare the following three lemmas:

LEMMA 1. *Let $\epsilon_D = \frac{1}{2}(t_0 + u_0\sqrt{D})$ be the fundamental unit of a real quadratic field $\mathbf{Q}(\sqrt{D})$ ($D > 0$ square-free). Then, u_0 is equal to 1 or 2 if and only if there exists an integer n such that $D = n^2 \pm 4$ or $n^2 \pm 1$.*

Proof. If there exists an integer n such that $D = n^2 \pm 4$ or $n^2 \pm 1$, then the result of Richaud-Degert[2] yields that the fundamental unit of $\mathbf{Q}(\sqrt{D})$ is $\epsilon_D = \frac{1}{2}(n + \sqrt{D})$ or $n + \sqrt{D}$, which implies $u_0 = 1$ or 2.

Conversely, since $t_0^2 - Du_0^2 = \mp 4$ holds, the assumption $u_0 = 1$ implies $D = t_0^2 \pm 4$, and the assumption $u_0 = 2$ implies $t_0 \equiv 0 \pmod 2$. Hence, if we put $t_0 = 2n$, then we get $n^2 - D = \mp 1$, which implies $D = n^2 \pm 1$.

Such type of real quadratic fields that the assumption of this lemma is satisfied, we call simply *R-D* type in narrow sense.

Remark. In the case of real quadratic fields with the fundamental unit whose norm is equal to -1, *R-D* type in narrow sense is the same as *R-D* type in wide sense defined in [4]. On the other hand, in the case of real quadratic fields with the fundamental unit whose norm is equal to

[1] Cf. H. Yokoi [4], Lemma 1.

[2] Cf. The Lemma 2 of H. Yokoi [4], H. Hasse [2] or G. Degert [1] and C. Richaud [3], etc.

1, *R-D* type in wide sense is the more general type than *R-D* type in narrow sense, but it is convenient for us to distinguish *R-D* type in narrow sense from the general type of Richaud-Degert in such a way.

LEMMA 2. *Let* $\epsilon_D = \frac{1}{2}(t_0 + u_0\sqrt{D})$ *(*$D > 0$ *square-free) be the fundamental unit of a real quadratic field* $\mathbf{Q}(\sqrt{D})$*, and let* $\epsilon = \frac{1}{2}(t + u\sqrt{D}) > 1$ *be any unit of* $\mathbf{Q}(\sqrt{D})$*. Then* $u \equiv 0 \ (\mathrm{mod}\ u_0)$ *holds.*

Proof. Let n be the positive integer satisfying $\epsilon = \epsilon_D{}^n$. Then, we may put

$$2^{n-1}(t + u\sqrt{D}) = 2^n\epsilon = (2\epsilon_D)^n = (t_0 + u_0\sqrt{D})^n = T + U\sqrt{D},$$

and get

$$U = 2^{n-1}u = \begin{cases} {}_nC_1 t_0^{n-1}u_0 + {}_nC_3 t_0^{n-3}u_0{}^3 D + \cdots + {}_nC_n u_0{}^n D^{(n-1/2)} & \text{for odd } n, \\ {}_nC_1 t_0^{n-1}u_0 + {}_nC_3 t_0^{n-3}u_0{}^3 D + \cdots + {}_nC_{n-1} t_0 u_0^{n-1} D^{(n-2/2)} & \text{for even } n. \end{cases}$$

Hence, we obtain $U = 2^{n-1}u \equiv 0 \ (\mathrm{mod}\ u_0)$. Therefore, in the case of $(u_0, 2) = 1$, we have $u \equiv 0 \ (\mathrm{mod}\ u_0)$.

On the other hand, in the case of $(u_0, 2) \neq 1$, we may put $t_0 = 2\bar{t}_0$, $u_0 = 2\bar{u}_0$ because of $t_0 \equiv u_0 \equiv 0 \ (\mathrm{mod}\ 2)$, and get $\epsilon_D = \bar{t}_0 + \bar{u}_0\sqrt{D}$. Since $\epsilon = \frac{1}{2}(t + u\sqrt{D}) = (\bar{t}_0 + \bar{u}_0\sqrt{D})^n$, we have

$$\tfrac{1}{2}u = \begin{cases} {}_nC_1 \bar{t}_0^{n-1}\bar{u}_0 + {}_nC_3 \bar{t}_0^{n-3}\bar{u}_0{}^3 D + \cdots + {}_nC_n \bar{u}_0{}^n D^{(n-1/2)} & \text{for odd } n, \\ {}_nC_1 \bar{t}_0^{n-1}\bar{u}_0 + {}_nC_3 \bar{t}_0^{n-3}\bar{u}_0{}^3 D + \cdots + {}_nC_{n-1} \bar{t}_0 \bar{u}_0^{n-1} D^{(n-2/2)} & \text{for even } n. \end{cases}$$

Hence, we obtain $\frac{1}{2}u \equiv 0 \ (\mathrm{mod}\ \bar{u}_0)$, which implies $u \equiv 0 \ (\mathrm{mod}\ 2\bar{u}_0 = u_0)$.

LEMMA 3. *Let* p *be any prime congruent to* -1 *mod* 4*, and assume that an unit* ϵ *of a real quadratic field* $\mathbf{Q}(\sqrt{D})$ *(*$D > 0$ *square-free) is of the form*

$$\epsilon = \tfrac{1}{2}(t + p\sqrt{D}) \qquad or \qquad t + p\sqrt{D} \quad (t > 0).$$

Then, the real quadratic field $\mathbf{Q}(\sqrt{D})$ *is of R-D type in narrow sense or the unit* ϵ *is the fundamental unit of* $\mathbf{Q}(\sqrt{D})$ *satisfying* $N\epsilon = 1$.

Proof. Let $\epsilon_D = \frac{1}{2}(t_0 + u_0\sqrt{D})$ be the fundamental unit of $\mathbf{Q}(\sqrt{D})$, then by Lemma 2 u_0 is equal to $1, 2, p$ or $2p$. In the case of $u_0 = 1$ or 2, by Lemma 1 the real quadratic field $\mathbf{Q}(\sqrt{D})$ is of *R-D* type in narrow sense.

In the case of $u_0 = p$, since $\epsilon_D = \frac{1}{2}(t_0 + p\sqrt{D})$ and $p \equiv -1 \ (\mathrm{mod}\ 4)$, by Lemma 1 of [4] we know $N\epsilon_D = 1$, and hence $N\epsilon = 1$. Therefore, for $\epsilon = \frac{1}{2}(t + p\sqrt{D})$ $t_0{}^2 - Dp^2 = 4N\epsilon_D = 4N\epsilon = t^2 - Dp^2$ implies $t = t_0$, which shows that the unit ϵ is equal to the fundamental unit ϵ_D. For $\epsilon = t + p\sqrt{D}$, there exists the positive integer $n > 1$ satisfying $\epsilon = \epsilon_D{}^n$, we have

$$2^n p = \begin{cases} ({}_nC_1 t_0^{n-1} + {}_nC_3 t_0^{n-3} p^2 D + \cdots + {}_nC_n p^{n-1} D^{(n-1/2)})p & \text{for odd } n, \\ ({}_nC_1 t_0^{n-1} + {}_nC_3 t_0^{n-3} p^2 D + \cdots + {}_nC_{n-1} t_0 p^{n-2} D^{(n-2/2)})p & \text{for even } n. \end{cases}$$

Therefore, for odd n we have $n \geqslant 3$, $p > 2$ and $D \geqslant 2$; hence, $2^n > {}_nC_n p^{n-1} D^{(n-1/2)} > 2^{n-1}2 = 2^n$ holds, which is a contradiction. For even n we have $2^n \equiv 0 \pmod{t_0}$, and hence $t_0 = 1$ or $t_0 \equiv 0 \pmod 2$ holds. However, $t_0 = 1$ implies $4 = t_0{}^2 - Dp^2 = 1 - Dp^2$, i.e., $Dp^2 = -3$, which is a contradiction. $t_0 \equiv 0 \pmod 2$ implies $u_0 \equiv t_0 \equiv 0 \pmod 2$, which conflicts with $u_0 = p \equiv -1 \pmod 4$.

In the case of $u_0 = 2p$, since $t_0 \equiv u_0 \equiv 0 \pmod 2$, we may put $t_0 = 2\bar{t}_0$, and get $\epsilon_D = \bar{t}_0 + p\sqrt{D}$. On the other hand, since $p \equiv -1 \pmod 4$, we know $N\epsilon_D = 1$, and hence $N\epsilon = 1$. Therefore, for $\epsilon = t + p\sqrt{D}$ we have $\bar{t}_0{}^2 - Dp^2 = N\epsilon_D = N\epsilon = t^2 - Dp^2$, i.e., $\bar{t}_0 = t$, which shows that the unit ϵ is equal to the fundamental unit ϵ_D. In this case of $u_0 = 2p$, since $\epsilon_D = \bar{t}_0 + p\sqrt{D}$, any unit $\epsilon > 1$ of $\mathbf{Q}(\sqrt{D})$ is never of the form $\frac{1}{2}(t + p\sqrt{D})$. Hence our lemma is proved in all cases.

THEOREM 1. *For any prime p congruent to -1 mod 4, there exists an integer $D_0 = D_0(p)$ such that if $D = p^2m^2 \pm 4m$ has no square factor except 4 and is bigger than D_0, then the fundamental unit ϵ_D of the real quadratic field $\mathbf{Q}(\sqrt{D})$ is of the following form:*

$$\epsilon_D = \begin{cases} \frac{1}{2}[(p^2m \pm 2) + p\sqrt{p^2m^2 \pm 4m}] & D\text{: square-free}, \\ \frac{1}{2}(p^2m \pm 2) + p\sqrt{\frac{1}{4}(p^2m^2 \pm 4m)} & \text{otherwise}, \end{cases}$$

and $N\epsilon_D = 1$.

Proof. In the case of $D = p^2m^2 \pm 4m$ square-free, since Pell's equation $t^2 - Du^2 = 4$ is satisfied by $D = p^2m^2 \pm 4m$, $t = p^2m \pm 2$, $u = p$, $\epsilon_0 = \frac{1}{2}[(p^2m \pm 2) + p\sqrt{p^2m^2 \pm 4m}]$ is an unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$. On the other hand, since $4 \not\equiv 0 \pmod p$, it follows from Lemma 5 of [4] that both $D \pm 1 = p^2m^2 \pm 4m \pm 1$ and $D \pm 4 = p^2m^2 \pm 4m \pm 4$ are never square for any natural m except at most a finite number, and hence by Lemma 2 of [4] the quadratic field $\mathbf{Q}(\sqrt{D})$ is of *R-D* type in narrow sense for at most a finite number of natural m. Therefore, by Lemma 3 the unit $\epsilon_0 = \frac{1}{2}[(p^2m \pm 2) + p\sqrt{p^2m^2 \pm 4m}]$ of the real quadratic field $\mathbf{Q}(\sqrt{D})$ is the fundamental unit for any square-free $D = p^2m^2 \pm 4m$ except at most a finite number, and $N\epsilon_0 = 1$ holds.

In the case that $D = p^2m^2 \pm 4m \equiv 0 \pmod 4$ and $D/4$ is square-free by the assumption $p \equiv -1 \pmod 4$, we have $m \equiv 0 \pmod 2$; hence, $p^2m \pm 2 \equiv 0 \pmod 2$. Therefore, if we put $m = 2m_0$, $t_0 = p^2m_0 \pm 1$, $\bar{D} = p^2m_0^2 \pm 2m_0$, then $\bar{D} = D/4$ is square-free and $t_0^2 - Dp^2 = 1$, and hence $\epsilon_0 = t_0 + p\sqrt{\bar{D}} = (p^2m_0 \pm 1) + p\sqrt{p^2m_0^2 \pm 2m_0}$ is an unit of the real quadratic field $\mathbf{Q}(\sqrt{\bar{D}})$. On the other hand, since $2 \not\equiv 0 \pmod p$, it follows from Lemma 5 of [4] that both $D \pm 1 = p^2m_0^2 \pm 2m_0 \pm 1$ and $D \pm 4 = p^2m_0^2 \pm 2m_0 \pm 4$ are never square for any natural m_0 except at most a finite number, and hence by Lemma 2 of [4] the real quadratic field $\mathbf{Q}(\sqrt{\bar{D}})$ is of *R-D* type in narrow sense for at most a finite number of natural m_0. Therefore, by Lemma 3 the unit

$$\epsilon_0 = (p^2m_0 \pm 1) + p\sqrt{p^2m_0^2 \pm 2m_0}$$

of the real quadratic field $\mathbf{Q}(\sqrt{\bar{D}})$ is the fundamental unit for any square-free $\bar{D} = p^2m_0^2 \pm 2m_0$ except at most a finite number, and $N\epsilon_0 = 1$ holds. The theorem is thus proved.

This theorem implies immediately the following sufficient condition for an unit ϵ with $N\epsilon = 1$ to be the fundamental unit of an real quadratic field $\mathbf{Q}(\sqrt{D})$:

COROLLARY. *For any prime p congruent to* -1 *mod* 4, *there exists an integer* $D_0 = D_0(p)$ *such that if for some square-free D bigger than* D_0 *the real quadratic field* $\mathbf{Q}(\sqrt{D})$ *contains an unit* ϵ *of the form*

$$\epsilon = \tfrac{1}{2}(t + p\sqrt{D}) \quad \text{or} \quad t + p\sqrt{D} \quad (t > 0),$$

then the unit ϵ *is the fundamental unit of* $\mathbf{Q}(\sqrt{D})$ *and* $N\epsilon = 1$ *holds.*

Proof. In the case of $\epsilon = \frac{1}{2}(t + p\sqrt{D})$, since the assumption $p \equiv -1 \pmod 4$ implies $N\epsilon = 1$, we have $t^2 - Dp^2 = 4$. Hence, $t^2 \equiv 4 \pmod{p^2}$ holds, which implies moreover $t \equiv \pm 2 \pmod{p^2}$. Therefore, for the integer $m_1 > 0$ satisfying $t = p^2m_1 \pm 2$ we get $Dp^2 = t^2 - 4 = p^2(p^2m_1^2 \pm 4m_1)$, i.e., $D = p^2m_1^2 \pm 4m_1$ and $\epsilon = \frac{1}{2}[(p^2m_1 \pm 2) + p\sqrt{p^2m_1^2 \pm 4m_1}]$ holds.

In the case of $\epsilon = t + p\sqrt{D}$, since we know similarly $N\epsilon = 1$, we get $t^2 - Dp^2 = 1$. Hence, $t^2 \equiv 1 \pmod{p^2}$ holds, which implies moreover $t \equiv \pm 1 \pmod{p^2}$. Therefore, for the integer $m_2 \geqslant 0$ satisfying $t = p^2m_2 \pm 1$ we have $Dp^2 = t^2 - 1 = p^2(p^2m_2^2 \pm 2m_2)$, i.e., $D = p^2m_2^2 \pm 2m_2$ and $\epsilon = (p^2m_2 \pm 1) + p\sqrt{p^2m_2^2 \pm 2m_2}$ holds. Hence, in both cases, if we choose D_0 in Theorem 1 as $D_0 = D_0(p)$ in question and consider square-free D bigger than $D_0(p)$, then it follows from Theorem 1 that the unit ϵ is the fundamental unit of $\mathbf{Q}(\sqrt{D})$.

2. The Class Number

In this section, we give an estimation from below of the class number of real quadratic fields considered in Section 1.

THEOREM 2. *For any prime p congruent to -1 mod 4, let $D_0 = D_0(p)$ be the integer in Theorem 1. Furthermore, set $D = p^2m^2 \pm 4m$ for any integer m bigger than $4p$, and consider D bigger than $D_0(p)$. Then, if D has no square factor except 4 and prime p splits in the real quadratic field $\mathbf{Q}(\sqrt{D})$ into two conjugate prime ideals with the degree one, these prime ideals are not principal. Therefore, the class number h of $\mathbf{Q}(\sqrt{D})$ is bigger than one and the following estimation from below holds*:

$$\begin{cases} h \geqslant \dfrac{\log(\sqrt{Dp^2+4}-2)}{\log p} - 2 & D\text{: square-free,} \\[2ex] h \geqslant \dfrac{\log(\frac{1}{4}\sqrt{Dp^2+4}-2)}{\log p} - 2 & \text{otherwise.} \end{cases}$$

Proof. In the case that $D = p^2m^2 \pm 4m$ is square-free, by Theorem 1 the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ is

$$\epsilon_D = \tfrac{1}{2}[(p^2m \pm 2) + p\sqrt{p^2m^2 \pm 4m}]$$

provided $D > D_0(p)$ and $N\epsilon = 1$ holds. Hence, it follows from Lemma 6 of [4] that Hasse's boundary s is

$$s = \frac{t_0 - 2}{u_0^2} = \frac{p^2m \pm 2 - 2}{p^2} = \begin{cases} m & \text{for + sign,} \\ m - \dfrac{4}{p^2} & \text{for − sign,} \end{cases}$$

and $0 < 4/p^2 < 1$. In the case that $D = p^2m^2 \pm 4m \equiv 0 \pmod 4$ and $D/4$ is square-free, by Theorem 1 the fundamental unit of $\mathbf{Q}(\sqrt{D})$ is $\epsilon_D = \frac{1}{2}(p^2m \pm 2) + p\sqrt{\frac{1}{4}(p^2m^2 \pm 4m)}$ provided $D > D_0(p)$ and $N\epsilon = 1$ holds. Hence, by Lemma 6 of [4] Hasse's boundary s is

$$s = \frac{t_0 - 2}{u_0^2} = \frac{p^2m \pm 2 - 2}{4p^2} = \begin{cases} \dfrac{m}{4} & \text{for + sign,} \\[2ex] \dfrac{m}{4} - \dfrac{1}{p^2} & \text{for − sign,} \end{cases}$$

and $0 < 1/p^2 < 1$. Therefore, for any integer m bigger than p (in the first case) or $4p$ (in the second case), the prime p is smaller than Hasse's boundary s, i.e., $p < s$.

If we assume that the prime p splits into two conjugate principal ideals $\mathfrak{p}$, $\mathfrak{p}'$ with the degree one in $\mathbf{Q}(\sqrt{D})$, then Pell's equation $\frac{1}{4}(x^2 - Dy^2) = \pm p$ is solvable, and hence Lemma 6 of [4] yields $p \geqslant s$, which conflicts with the above assertion $p < s$. Therefore, if the prime p splits into two conjugate prime ideals $\mathfrak{p}$, $\mathfrak{p}'$ with the degree one in $\mathbf{Q}(\sqrt{D})$, then the prime $\mathfrak{p}$, $\mathfrak{p}'$ are not principal. Moreover, the order of those prime ideals $\mathfrak{p}$, $\mathfrak{p}'$ in the ideal class group of $\mathbf{Q}(\sqrt{D})$ is bigger than one and it is a factor of the ideal class number h of $\mathbf{Q}(\sqrt{D})$. Hence, in the first case, we obtain

$$p^h = N\mathfrak{p}^h \geqslant s = \frac{p^2m \pm 2 - 2}{p^2} = \frac{\sqrt{Dp^2 + 4} - 2}{p^2},$$

which implies

$$h \geqslant \frac{\log(\sqrt{Dp^2 + 4} - 2)}{\log p} - 2.$$

Similarly, in the second case, we obtain

$$p^h = N\mathfrak{p}^h \geqslant s = \frac{p^2m \pm 2 - 2}{4p^2} = \frac{\sqrt{Dp^2 + 4} - 2}{4p^2},$$

which implies

$$h \geqslant \frac{\log \frac{1}{4}(\sqrt{Dp^2 + 4} - 2)}{\log p} - 2.$$

Thus, the theorem is completely proved.

References

1. G. Degert, Über die Bestimmung der Grundeinheit gewisser reell-quadratischer Zahlkörper, *Abh. math. Sem. Univ. Hamburg* **22** (1958), 92–97.
2. H. Hasse, Über mehrklassige, aber eingeschlechtige reell-quadratische Zahlkörper, *Elem. Math.* **20** (1965), 49–59.
3. C. Richaud, Sur la résolution des équations $x^2 - Ay^2 = +1$, *Atti Accad. pontif. Nuovi Lincei* (1866), 177–182.
4. H. Yokoi, On real quadratic fields containing units with norm -1, *Nagoya Math. J.* **33** (1968), 139–152.

TABLE I

THE CASE OF $p = 3$

t	D	u	m	t	D	u
	$t = 9m - 2$, $D = 9m^2 - 4m$				$t = 9m + 2$, $D = 9m^2 + 4m$	
7	$5 ; \epsilon_5^4$*	3	1	11	$13 ; \epsilon_{13}^2$	3
16	7	6	2	20	11	6
25	$69 = 3 \cdot 23$	3	3	29	$93 = 3 \cdot 31$	3
34	$2 ; \epsilon_2^4$	24	4	38	$10 = 2 \cdot 5; \epsilon_{10}^2$	12
43	$205 = 5 \cdot 41$	3	5	47	$5 ; \epsilon_5^8$	21
52	$3 ; \epsilon_3^3$	30	6	56	$87 = 3 \cdot 29$	6
61	$413 = 7 \cdot 59$	3	7	65	$469 = 7 \cdot 67$	3
70	$34 = 2 \cdot 17$	12	8	74	$38 = 2 \cdot 19$	12
79	$77 = 7 \cdot 11 ; \epsilon_{77}^2$	9	9	83	$85 = 5 \cdot 17 ; \epsilon_{85}^2$	9
88	$215 = 5 \cdot 43$	6	10	92	$235 = 5 \cdot 47$	6
97	$1045 = 5 \cdot 11 \cdot 19$	3	11	101	$1133 = 11 \cdot 103$	3
106	$78 = 2 \cdot 3 \cdot 13$	12	12	110	$21 = 3 \cdot 7 ; \epsilon_{21}^2$	24
115	$1469 = 13 \cdot 113$	3	13	119	$13 ; \epsilon_{13}^4$	33
124	$427 = 7 \cdot 61$	6	14	128	$455 = 5 \cdot 7 \cdot 13$	6
133	$1965 = 3 \cdot 5 \cdot 131$	3	15	137	$2085 = 3 \cdot 5 \cdot 139$	3
142	$35 = 5 \cdot 7 ; \epsilon_{35}^2$	24	16	146	$37 ; \epsilon_{37}^2$	24
151	$2533 = 17 \cdot 149$	3	17	155	$2669 = 17 \cdot 157$	3
160	79	18	18	164	83	18
169	$3173 = 19 \cdot 167$	3	19	173	$133 = 7 \cdot 19$	15
178	$55 = 5 \cdot 11$	24	20	182	$230 = 2 \cdot 5 \cdot 23$	12
187	$3885 = 3 \cdot 5 \cdot 7 \cdot 37$	3	21	191	$4053 = 3 \cdot 7 \cdot 193$	3
196	$1067 = 11 \cdot 97$	6	22	200	$1111 = 11 \cdot 101$	6
205	$4669 = 7 \cdot 23 \cdot 29$	3	23	209	$4853 = 23 \cdot 211$	3
214	$318 = 2 \cdot 3 \cdot 53$	12	24	218	$330 = 2 \cdot 3 \cdot 5 \cdot 11$	12
223	$221 = 13 \cdot 17 ; \epsilon_{221}^2$	15	25	227	$229 ; \epsilon_{229}^2$	15
232	$1495 = 5 \cdot 13 \cdot 23$	6	26	236	$1547 = 7 \cdot 13 \cdot 17$	6
241	$717 = 3 \cdot 239$	9	27	245	$741 = 3 \cdot 13 \cdot 19$	9
250	$434 = 2 \cdot 7 \cdot 31$	12	28	254	$7 ; \epsilon_7^2$	96
259	$7453 = 29 \cdot 257$	3	29	263	$7685 = 5 \cdot 29 \cdot 53$	3
268	$1995 = 3 \cdot 5 \cdot 7 \cdot 19$	6	30	272	$2055 = 3 \cdot 5 \cdot 137$	6

* ϵ_5^4 means the fourth power of the fundamental unit ϵ_5 of the real quadratic field $\mathbf{Q}(\sqrt{5})$, etc.

TABLE II

The Case of $p = 7$

	$t = 49m - 2$				$t = 49m + 2$	
	$D = 49m^2 - 4m$				$D = 49m^2 + 4m$	
t	D	u	m	t	D	u
47	$5\ ;\ \epsilon_5^8$	21	1	51	$53\ ;\ \epsilon_{53}^2$	7
96	47	14	2	100	51	14
145	$429 = 3 \cdot 11 \cdot 13$	7	3	149	$453 = 3 \cdot 151$	7
194	$3\ ;\ \epsilon_3^4$	112	4	198	$2\ ;\ \epsilon_2^6$	140
243	$1205 = 5 \cdot 241$	7	5	247	$1245 = 3 \cdot 5 \cdot 83$	7
292	$435 = 3 \cdot 5 \cdot 29$	14	6	296	$447 = 3 \cdot 149$	14
341	$2373 = 3 \cdot 7 \cdot 113$	7	7	345	$2429 = 7 \cdot 347$	7
390	$194 = 2 \cdot 97$	28	8	394	$22 = 2 \cdot 11$	84
439	$437 = 19 \cdot 23$	21	9	443	$445 = 5 \cdot 89\ ;\ \epsilon_{445}^2$	21
488	$15 = 3 \cdot 5\ ;\ \epsilon_{15}^3$	126	10	492	$1235 = 5 \cdot 13 \cdot 19$	14
537	$5885 = 5 \cdot 11 \cdot 107$	7	11	541	$5973 = 3 \cdot 11 \cdot 181$	7
586	$438 = 2 \cdot 3 \cdot 73$	28	12	590	$111 = 3 \cdot 37$	56
635	$8229 = 3 \cdot 13 \cdot 211$	7	13	639	$8333 = 13 \cdot 641$	7
684	$2387 = 7 \cdot 11 \cdot 31$	14	14	688	$2415 = 3 \cdot 5 \cdot 7 \cdot 23$	14
733	$10965 = 3 \cdot 5 \cdot 17 \cdot 43$	7	15	737	$11085 = 3 \cdot 5 \cdot 739$	7
782	$195 = 3 \cdot 5 \cdot 13\ ;\ \epsilon_{195}^2$	56	16	786	$197\ ;\ \epsilon_{197}^2$	56
831	$14093 = 17 \cdot 829$	7	17	835	$1581 = 17 \cdot 93$	21
880	439	42	18	884	443	42
929	$1957 = 19 \cdot 103$	21	19	933	$17765 = 5 \cdot 11 \cdot 17 \cdot 19$	7
978	$305 = 5 \cdot 61$	56	20	982	$1230 = 2 \cdot 3 \cdot 5 \cdot 41$	28
1027	$861 = 3 \cdot 7 \cdot 41$	35	21	1031	$21693 = 3 \cdot 7 \cdot 1033$	7
1076	$5907 = 3 \cdot 11 \cdot 179$	14	22	1080	$5951 = 11 \cdot 541$	14
1125	$25829 = 23 \cdot 1123$	7	23	1129	$26013 = 3 \cdot 13 \cdot 23 \cdot 29$	7
1174	$1758 = 2 \cdot 3 \cdot 293$	28	24	1178	$1770 = 2 \cdot 3 \cdot 5 \cdot 59$	28
1223	$1221 = 3 \cdot 11 \cdot 37\ ;\ \epsilon_{1221}^2$	35	25	1227	$1229\ ;\ \epsilon_{1229}^2$	35
1272	$8255 = 5 \cdot 13 \cdot 127$	14	26	1276	$923 = 13 \cdot 71$	42
1321	$3957 = 3 \cdot 1319$	21	27	1325	$3981 = 3 \cdot 1327$	21
1370	$266 = 2 \cdot 7 \cdot 19$	84	28	1374	$602 = 2 \cdot 7 \cdot 43$	56
1419	$41093 = 13 \cdot 29 \cdot 109$	7	29	1423	$1653 = 3 \cdot 19 \cdot 29$	35
1468	$10995 = 3 \cdot 5 \cdot 733$	14	30	1472	$11055 = 3 \cdot 5 \cdot 11 \cdot 67$	14

TABLE 3

The Case of $p = 11$

$t = 121m - 2$	$D = 121m^2 - 4m$			$t = 121m + 2$	$D = 121m^2 + 4m$	
t	D	u	m	t	D	u
119	$13 ; \epsilon_{13}^4$	33	1	123	$5 ; \epsilon_5^{10}$	55
240	$119 = 7 \cdot 17$	22	2	244	$123 = 3 \cdot 41$	22
361	$1077 = 3 \cdot 359$	11	3	365	$1101 = 3 \cdot 367$	11
482	$30 = 2 \cdot 3 \cdot 5 ; \epsilon_{30}^2$	88	4	486	$122 = 2 \cdot 61 ; \epsilon_{122}^2$	44
603	$3005 = 5 \cdot 601$	11	5	607	$3045 = 3 \cdot 5 \cdot 7 \cdot 29$	11
724	$3 ; \epsilon_3^5$	418	6	728	$1095 = 3 \cdot 5 \cdot 73$	22
845	$5901 = 3 \cdot 7 \cdot 281$	11	7	849	$5957 = 7 \cdot 23 \cdot 37$	11
966	$482 = 2 \cdot 241$	44	8	970	$6 = 2 \cdot 3; \epsilon_6^3$	396
1087	$1085 = 5 \cdot 7 \cdot 31 ; \epsilon_{1085}^2$	33	9	1091	$1093 ; \epsilon_{1093}^2$	33
1208	$335 = 5 \cdot 67$	66	10	1212	$3035 = 5 \cdot 607$	22
1329	$14597 = 11 \cdot 1327$	11	11	1333	$14685 = 3 \cdot 5 \cdot 11 \cdot 89$	11
1450	$1086 = 2 \cdot 3 \cdot 181$	44	12	1454	$273 = 3 \cdot 7 \cdot 13$	88
1571	$20397 = 3 \cdot 13 \cdot 523$	11	13	1575	$20501 = 13 \cdot 19 \cdot 83$	11
1692	$35 = 5 \cdot 7 ; \epsilon_{35}^3$	286	14	1696	$5943 = 3 \cdot 7 \cdot 283$	22
1813	$27165 = 3 \cdot 5 \cdot 1811$	11	15	1817	$27285 = 3 \cdot 5 \cdot 17 \cdot 107$	11
1934	$483 = 3 \cdot 7 \cdot 23 ; \epsilon_{483}^2$	88	16	1938	$485 = 5 \cdot 97 ; \epsilon_{485}^2$	88
2055	$34901 = 17 \cdot 2053$	11	17	2059	$3893 = 17 \cdot 229$	33
2176	1087	66	18	2180	1091	66
2297	$4845 = 3 \cdot 5 \cdot 17 \cdot 19$	33	19	2301	$893 = 19 \cdot 47$	77
2418	$755 = 5 \cdot 151$	88	20	2422	$3030 = 2 \cdot 3 \cdot 5 \cdot 101$	44
2539	$53277 = 3 \cdot 7 \cdot 43 \cdot 59$	11	21	2543	$53445 = 3 \cdot 5 \cdot 7 \cdot 509$	11
2660	$14619 = 3 \cdot 11 \cdot 443$	22	22	2664	$14663 = 11 \cdot 31 \cdot 43$	22
2781	$63917 = 7 \cdot 23 \cdot 397$	11	23	2785	$64101 = 3 \cdot 23 \cdot 929$	11
2902	$174 = 2 \cdot 3 \cdot 29$	220	24	2906	$4362 = 2 \cdot 3 \cdot 727$	44
3023	$3021 = 3 \cdot 19 \cdot 53 ; \epsilon_{3021}^2$	55	25	3027	$3029 = 13 \cdot 233 ; \epsilon_{3029}^2$	55
3144	$20423 = 13 \cdot 1571$	22	26	3148	$819 = 3 \cdot 13 \cdot 21$	110
3265	$9789 = 3 \cdot 13 \cdot 251$	33	27	3269	$9813 = 3 \cdot 3271$	33
3386	$658 = 2 \cdot 7 \cdot 47$	132	28	3390	$371 = 7 \cdot 53$	176
3507	$101645 = 5 \cdot 29 \cdot 701$	11	29	3511	$101877 = 3 \cdot 29 \cdot 1171$	11
3628	$555 = 3 \cdot 5 \cdot 37$	154	30	3632	$27255 = 3 \cdot 5 \cdot 23 \cdot 79$	22